

SYZYGY ALGEBRAS FOR THE SEGRE EMBEDDING

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ABSTRACT. We describe the syzygy algebra for the Segre embedding in terms of representations of $\mathrm{GL}(V)$.

1. INTRODUCTION

In the paper [1] the syzygy algebras of the grassmannians $\mathrm{Gr}(2, N)$ are found. Using slightly different methods we will find syzygy algebra for the Segre embedding $\mathbb{P}^{\hat{n}} \times \mathbb{P}^{\hat{n}} \hookrightarrow \mathbb{P}^{\hat{n}\hat{n}+\hat{n}+\hat{n}}$ (see Theorems (4.1) and (5.1)).

The paper is organized as follows. In the second section we follow [1] and [2] and describe Koszul complex such that its cohomology groups equal syzygy spaces. In the third section we describe the projective coordinate algebras of the highest weight orbits. In the next one we calculate the syzygy spaces of the Segre embedding and in the last section we describe multiplication in this syzygy algebra.

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2. KOSZUL COMPLEX

Let $\varphi: X \rightarrow \mathbb{P}(W)$ be an embedding of a projective variety X into a projective space. Denote by $S = S^\bullet(W^*)$ the projective coordinate algebra of $\mathbb{P}(W)$ and by A the projective coordinate algebra of X . Consider the *minimal free resolution*

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A$$

which is an exact sequence of graded free S -modules of the form

$$F_p = \bigoplus_{q \geq m_p} R_{p,q} \otimes_{\mathbb{C}} S[-q],$$

where $R_{p,q}$ are finite dimensional vector spaces of p -th order syzygies of degree q . The minimality of the resolution means that all homogeneous components of the differential

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have positive degrees. Thus, tensor multiplication by the trivial S -module \mathbb{C} annihilates all differentials in the minimal free resolution and we get

$$(2.1) \quad R_{p,q} = (\mathrm{Tor}_p^S(A, \mathbb{C}))_q.$$

Syzygy spaces may be found by tensoring the Koszul complex (for example, see [2])

$$\dots \xrightarrow{d} K_2 \xrightarrow{d} K_1 \xrightarrow{d} K_0 \xrightarrow{d} \mathbb{C} \rightarrow 0$$

for the trivial S -module \mathbb{C} by A , where $K_i = \Lambda^i W^* \otimes S[-i]$:

$$(2.2) \quad 0 \rightarrow \Lambda^n W^* \otimes_{\mathbb{C}} A[-n] \rightarrow \dots \rightarrow W^* \otimes_{\mathbb{C}} A[-1] \rightarrow A \rightarrow 0.$$

Since the differential is homogeneous, we can decompose the complex (2.2):

$$\dots \longrightarrow \Lambda^{p+1} W^* \otimes A_{q-1} \xrightarrow{d_{p-1,q+1}} \Lambda^p W^* \otimes A_q \xrightarrow{d_{p,q}} \Lambda^{p-1} W^* \otimes A_{q+1} \longrightarrow \dots$$

The identity element of $W \otimes W^*$ induces the natural map $\iota: \Lambda^p W^* \rightarrow \Lambda^{p-1} W^* \otimes W^*$ dual to the exterior product map $\Lambda^{p-1} W \otimes W \rightarrow \Lambda^p W$. Given the multiplication $m_q: W^* \otimes A_q \rightarrow A_{q+1}$, we define $d_{p,q}$ as the composition:

$$\begin{array}{ccc} \Lambda^p W^* \otimes A_q & \xrightarrow{\iota \otimes \mathrm{Id}} & \Lambda^{p-1} W^* \otimes W^* \otimes A_q \\ & \searrow d_{p,q} & \downarrow \mathrm{Id} \otimes m_q \\ & & \Lambda^{p-1} W^* \otimes A_{q+1} \end{array}$$

The *Koszul cohomology groups* are

$$\mathcal{K}_{p,q} = \frac{\ker(d_{p,q})}{\mathrm{im}(d_{p+1,q-1})}.$$

Finally, we get

$$(2.3) \quad R_{p,p+q} = (\mathrm{Tor}_p^S(A, \mathbb{C}))_{p+q} = \frac{\ker(d_{p,q})}{\mathrm{im}(d_{p+1,q-1})} = \mathcal{K}_{p,q}.$$

3. PROJECTIVE COORDINATE ALGEBRAS

Let G be a reductive algebraic group. Fix a maximal torus $T \subset G$ and any ordering on the roots. By $W = V_{-\lambda}$ denote the unique irreducible representation with the highest weight $-\lambda$. Let W be the orbit of the highest weight vector w of weight $-\lambda$.

Remark 3.1. *Remind that $G \cdot w = X = G/P$ is a projective variety.*

Proposition 3.2. *In the above notations, the projective coordinate algebra of X is*

$$\bigoplus_{n \geq 0} V_{n\lambda},$$

where $V_{n\lambda}$ is the irreducible representation with the highest weight $n\lambda$.

Proof. The projective coordinate algebra of $\mathbb{P}(W)$ equals

$$S = \bigoplus_{n \geq 0} S^n W = \bigoplus_{n \geq 0} S^n V_\lambda.$$

Let $w \in W$ be a vector of the highest weight. Let ω be a symmetrical form of degree n . Suppose that ω has the weight $n\lambda$. Then $\omega(G \cdot w) \neq 0$. Suppose that ω has a lower weight. Then

$$\begin{aligned} \omega(G \cdot w) &= \{\omega(gw) | g \in G\} = \{g^{-1}\omega | g \in G\}(w) = \\ &= (G \cdot \omega)(w) = \left\{ \sum_{\alpha < \lambda} \omega_\alpha(g) | g \in G \right\} (w) = 0, \end{aligned}$$

where $g\omega = \sum_{\alpha < \lambda} \omega_\alpha(g)$ is the weight decomposition. This implies that there is a unique irreducible component in A_n and it has the weight $n\lambda$. \square

Corollary 3.3. *In the above notation the complex of representations of G*

$$\dots \rightarrow \Lambda^{p+1} W^* \otimes_{\mathbb{C}} V_{(q-1)\lambda} \rightarrow \Lambda^p W^* \otimes_{\mathbb{C}} V_{q\lambda} \rightarrow \Lambda^{p-1} W^* \otimes_{\mathbb{C}} V_{(q+1)\lambda} \rightarrow \dots$$

gives the syzygy spaces of $X = G \cdot w \subset W$.

Remind that Σ_λ is a Schur functor and for $G = \mathrm{GL}(V)$ the representation $\Sigma_\lambda V$ is the unique representation with the highest weight λ .

Corollary 3.4. *Let $W_1 \otimes \dots \otimes W_m$ be an irreducible $\mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_m)$ -module and $X \subset \mathbb{P}(W)$ be a orbit of highest weight $\lambda_1 \oplus \dots \oplus \lambda_m$, where $W = W_1 \otimes \dots \otimes W_m$. Then the projective coordinate algebra of X equals*

$$A = \bigoplus_{n \geq 0} \Sigma_{-n\lambda_1} V_1 \otimes \dots \otimes \Sigma_{-n\lambda_m} V_m$$

as a $\mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_m)$ -module.

The next series of statements in this section is not needed in the rest of the paper, but we include it for its own interest.

Let A be a graded algebra. It is called *one-generated* if the natural map $T^\bullet(A_1) \rightarrow A$ is surjective. One-generated algebra A is called *quadratic* if its kernel J_A is generated as two-sided ideal in $T^\bullet(A_1)$ by its subspace $I_A = J_A \cap A_1 \subset A_1 \otimes A_1$. Denote by V the space A_1 , by Q the space $I_A \subset T^2(A_1)$ and by (V, Q) the algebra A . The *quadratic dual algebra* A^\perp is defined by the pair (V^*, Q^\perp) .

A quadratic algebra is called a *Koszul algebra* if the following pairwise equivalent conditions are satisfied.

- $A \simeq \mathrm{Ext}_{T(V^*)/(I^\perp)}^{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.

- $\text{Ext}_{T(V^*)/(I^\perp)}^{i,j}(\mathbb{C}, \mathbb{C}) = 0$ for $i \neq j$, where $\text{Ext}^{i,j}$ means the internal degree j graded component of Ext^i .
- for each $m \geq 3$ there exists a basis $E \subset V^{*\otimes n}$ for $V^{*\otimes n}$ such that $W_\alpha \cap E$ is a basis for W_α , where $W_\alpha = V^{*\otimes \alpha} \otimes Q^\perp \otimes V^{*\otimes(m-\alpha-2)}$.

A graded algebra A is called *monomial* if there exists a basis $E = \{e_i\}$ for A as a vector space such that $e_i \cdot e_j \in E$ for all i, j and each homogeneous component A_n is spanned by some subset of E . The following proposition is well known (see [4]), but we prove it for convinience of the reader.

Proposition 3.5. *Let $A = (V, Q)$ be a quadratic monomial algebra. Then A is a Koszul algebra.*

Proof. Let $V \cap E = \{v_1, \dots, v_n\}$, $\{v_1^* \dots, v_n^*\}$ be the dual basis for $\{v_1, \dots, v_n\}$, $Q = \langle \{q_1, \dots, q_N\} \rangle$, $q_i = v_{\alpha_i} \otimes v_{\beta_i}$. Hence, $Q^\perp = \langle v_\gamma^* \otimes v_\delta^* \rangle$, where $v_\gamma \otimes v_\delta \notin Q$. Therefore,

$$\begin{aligned} W_\nu &= V^{*\otimes \nu} \otimes Q^\perp \otimes V^{*\otimes m-\nu-2} = \\ &= \langle \{v_{\gamma_1}^* \otimes \dots \otimes v_{\gamma_m}^* | (\gamma_{\nu+1}, \gamma_{\nu+2}) \notin \{(i, j) | v_{\gamma_i} \otimes v_{\gamma_j} \in Q\}\} \rangle, \\ &\{v_{\gamma_1}^* \otimes \dots \otimes v_{\gamma_m}^* | (\gamma_{\nu+1}, \gamma_{\nu+2}) \notin \{(i, j) | v_{\gamma_i} \otimes v_{\gamma_j} \in Q\}\} \subset \{v_{\gamma_1}^* \otimes \dots \otimes v_{\gamma_m}^*\}, \\ &\langle \{v_{\gamma_1}^* \otimes \dots \otimes v_{\gamma_m}^*\} \rangle = V^{*\otimes n}. \end{aligned}$$

□

Proposition 3.6. *Let W be an irreducible representation of $\text{GL}(V)$ and $X \subset \mathbb{P}(W)$ be the orbit of highest weight λ . Then the projective coordinate algebra of X is monomial.*

Proof. The projective coordinate algebra of X equals $\bigoplus_{n \geq 0} \Sigma_{n\lambda} V$. Note that any irreducible component has a basis that consists of semistandard Young tableaux and we can multiply them using Littlewood–Richardson rule (see [3, Part 1, §1]). In the product of $\Sigma_{n\lambda} V$ and $\Sigma_{m\lambda} V$ all components except $\Sigma_{(m+n)\lambda} V$ are annihilated. These Young tableaux form a monomial basis for the algebra A . □

By the same reasons the projective coordinate algebra of highest weight orbit of $\text{GL}(V_1) \times \dots \times \text{GL}(V_n)$ in any irreducible representation is monomial as well. Below we will consider only orbits of this type. Therefore, all projective coordinate algebras in what follows will be Koszul algebras.

4. SYZGY SPACES

Let $\mathbb{P}^{\dot{n}} \times \mathbb{P}^{\dot{n}} \rightarrow \mathbb{P}^{\dot{n}\dot{n}+\dot{n}+\dot{n}}$ be the Segre embedding. It takes

$$((\dot{x}_0: \dots : \dot{x}_{\dot{n}}), (\dot{x}_0: \dots : \dot{x}_{\dot{n}})) \mapsto (\dot{x}_0\dot{x}_0: \dots : \dot{x}_i\dot{x}_j: \dots : \dot{x}_{\dot{n}}\dot{x}_{\dot{n}}).$$

Let us denote coordinates on $\mathbb{P}^{\dot{n}\dot{n}+\dot{n}+\dot{n}}$ by $\{x_{i,j}\}$ and the projective coordinate algebra of $\mathbb{P}^{\dot{n}\dot{n}+\dot{n}+\dot{n}}$ by S .

We have the group actions

$$\mathrm{GL}(\dot{n} + 1) = \dot{G}: \dot{S} = \mathbb{C}[\dot{x}, \dots, \dot{x}_{\dot{n}}],$$

$$\mathrm{GL}(\dot{n} + 1) = \dot{G}: \dot{S} = \mathbb{C}[\dot{x}, \dots, \dot{x}_{\dot{n}}].$$

Let us denote the dual to the tautological representations of the groups \dot{G} and \dot{G} by $\dot{V} = \dot{S}_1$ and $\dot{V} = \dot{S}_1$. The space $W = S_1 = \dot{V} \otimes \dot{V}$ is the representation of the group $G = \dot{G} \times \dot{G}$.

Note that the group $\dot{G} \times \dot{G}$ has the equivariant action on the graded algebra A and A_n equals $S^n(\dot{V}) \otimes S^n(\dot{V})$ as a G -module.

Let us fix some notation. Let T be a Young diagram. Denote by \tilde{T}_n the diagram the corresponds to the representation of the minimal weight in the decomposition of $\Sigma_T(V) \otimes S^n(V)$. We get \tilde{T}_n by adding a cell to the end of the first n columns of T . Denote by $\mathrm{wt}(T)$ the weight of T (i.e. number of cells), by $l(T)$ the length of diagonal of T and by T' the transposed diagram T .

Theorem 4.1. *There is a canonical isomorphism of representations of G :*

$$R_{p,q} = \bigoplus_{\substack{\mathrm{wt}(T)=p, \\ l(T)=q-p}} \left(\Sigma_{\tilde{T}_l(T)}(\dot{V}) \otimes \Sigma_{\tilde{T}'_l(T)}(\dot{V}) \right).$$

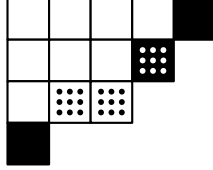
Proof. According to corollary (3.3) we need to calculate cohomology groups of the following complex of representation of G :

$$\begin{aligned} (4.1) \quad \dots \rightarrow \Lambda^{p+1}(\dot{V} \otimes \dot{V}) \otimes S^{q-1}(\dot{V}) \otimes S^{q-1}(\dot{V}) \rightarrow \\ \rightarrow \Lambda^p(\dot{V} \otimes \dot{V}) \otimes S^q(\dot{V}) \otimes S^q(\dot{V}) \rightarrow \\ \rightarrow \Lambda^{p-1}(\dot{V} \otimes \dot{V}) \otimes S^{q+1}(\dot{V}) \otimes S^{q+1}(\dot{V}) \rightarrow \dots \end{aligned}$$

We will decompose this complex into the sum of subcomplexes such that each subcomplex has equal irreducible components. Since there are no homomorphisms between these subcomplexes, we could calculate cohomology groups separately.

We have $\Lambda^k(\dot{V} \otimes \dot{V}) = \sum_{\lambda} \Sigma_{\lambda}(\dot{V}) \otimes \Sigma_{\lambda'}(\dot{V})$, where λ runs over all Young diagrams of the weight n . The irreducible G -modules are indexed by the pairs of Young diagrams. Let us transpose the first one and put over the second. Then fill the cells coming from $S^q(\dot{V})$ by black color, fill the cells coming from $S^q(\dot{V})$ by dots, and leave the cells coming

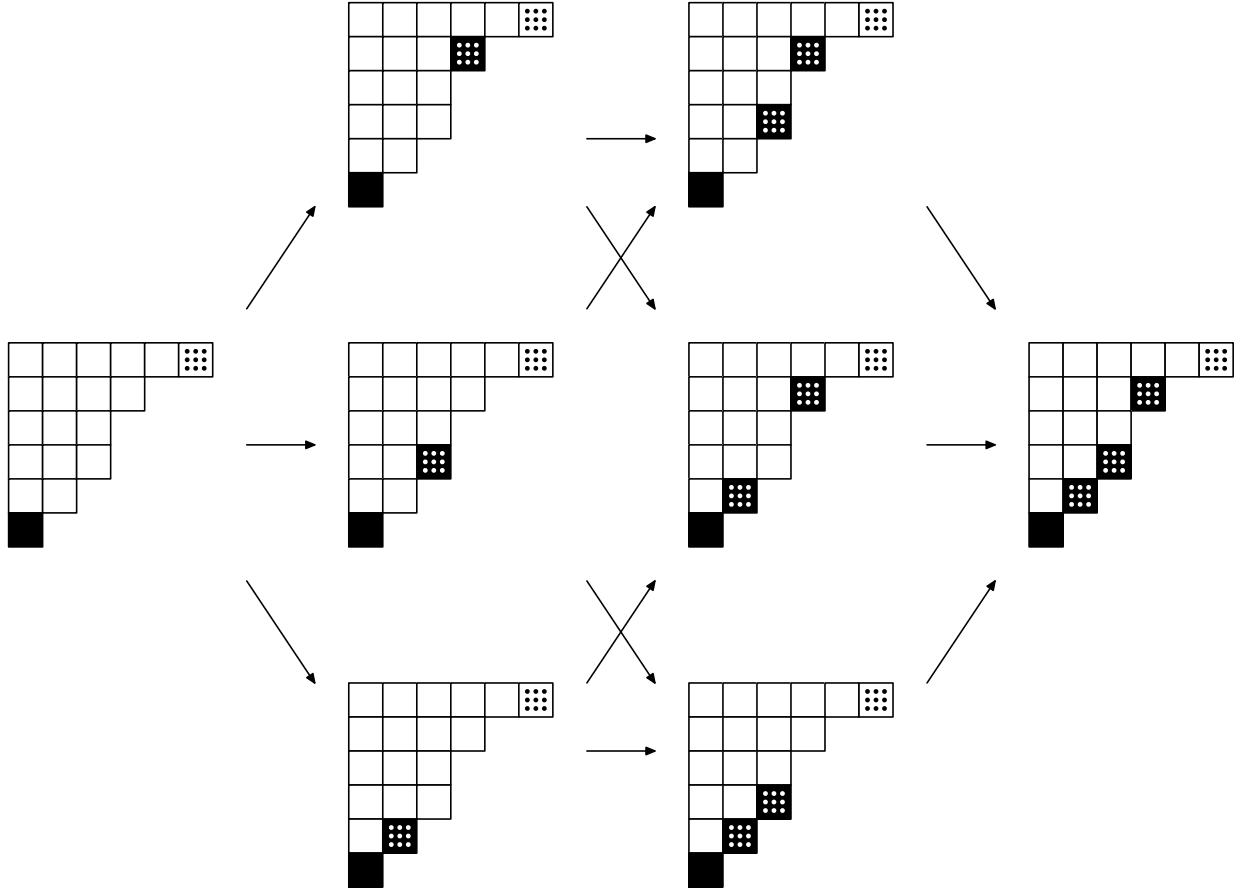
from $\Lambda^k(\dot{V} \otimes \dot{V})$ uncolored. Thus, we obtain a *colored diagram* that consists of a number of white cells and some cells filled by black, by dots and by the both black and dots. Note that the locus of white cells, the locus of either white or dotted cells, and the locus of either white or black cells are valid Young diagrams. Also, there are no two black cells in one row and there are no two dotted cells in one column. This looks like on the picture.



We will group diagrams into series. We put in the same series all the diagrams with

- the same set of cells,
- the same set of black but not dotted cells,
- the same set of dotted but not filled cells.

Note that the set of white cells and the set of dotted black cells may vary in the same series. Each G -module of Koszul complex corresponds to one colored diagram. The differential d preserves the series. The picture below demonstrates the action of the differential d on some series.



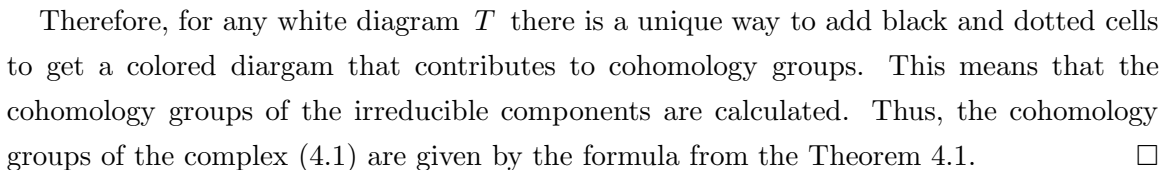
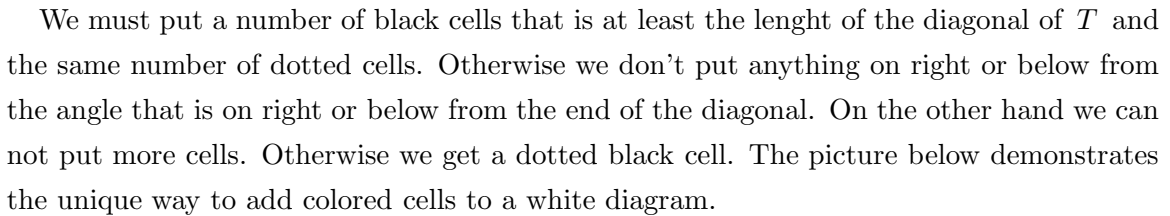
Thus, if a series consists of more than one diagram then it does not contribute to cohomology groups. We see that any diagram with a dotted black cell is included into a series that doesn't contribute to cohomology groups. Also, if we can substitute a white cell for a dotted black cell in some colored diagram, then these two colored diagrams are in the same series and don't contribute to cohomology groups.

Obviously, if the diagram has a dotted black cell then there are no dotted cells at the same row and there are no black cells at the same column. Also there are no dotted cells on the top, black cells on the left and white cells on the right or below, because otherwise dotted and white or black and white cells would not form a diagram. It means that the differential sends the G -module that corresponds to the diagram T to the sum of G -modules that correspond to diagrams T' obtained by replacement of a white cell in the diagram T in an *exterior angle*¹ by a black dotted cell.

Let us now describe all G -modules that contribute to cohomology groups. Each such a G -module is obtained by the following construction. Take a white Young diagram T that

¹such that there are no white cells on the right or below

Consider two neighbour exterior angles (the second one is on the up and right from the first). We can not put a black cell at the right of the first angle and a dotted cell below from the second. Otherwise we do not get a valid colored diagram or get a dotted black cell.



The multiplication on the syzygy algebra is induced by multiplication in DG -algebra $A \otimes \Lambda^\bullet(W)$,

$$m: (\Lambda^{p_1}(\dot{V} \otimes \dot{V}) \otimes S^{q_1} \dot{V} \otimes S^{q_1} \dot{V}) \otimes (\Lambda^{p_2}(\dot{V} \otimes \dot{V}) \otimes S^{q_2} \dot{V} \otimes S^{q_2} \dot{V}) \rightarrow \Lambda^{p_1+p_2}(\dot{V} \otimes \dot{V}) \otimes S^{q_1+q_2} \dot{V} \otimes S^{q_1+q_2} \dot{V}.$$

It comes from the natural multiplications

$$\begin{aligned} \Lambda^{p_1}(\dot{V} \otimes \dot{V}) \otimes \Lambda^{p_2}(\dot{V} \otimes \dot{V}) &\rightarrow \Lambda^{p_1+p_2}(\dot{V} \otimes \dot{V}), \\ S^{q_1}\dot{V} \otimes S^{q_2}\dot{V} &\rightarrow S^{q_1+q_2}\dot{V}, \quad S^{q_1}\dot{V} \otimes S^{q_2}\dot{V} \rightarrow S^{q_1+q_2}\dot{V}. \end{aligned}$$

The first one takes $(\Sigma_\lambda \dot{V} \otimes \Sigma_{\lambda'} \dot{V}) \otimes (\Sigma_\mu \dot{V} \otimes \Sigma_{\mu'} \dot{V}) \rightarrow \sum_{\nu \subset \lambda \otimes \mu} (\Sigma_\nu \dot{V} \otimes \Sigma_{\nu'} \dot{V})$.

In the cohomology algebra the summands

$$\Sigma_\nu \dot{V} \otimes \Sigma_{\nu'} \dot{V} \otimes S^{q_1+q_2}\dot{V} \otimes S^{q_1+q_2}\dot{V}$$

of the product

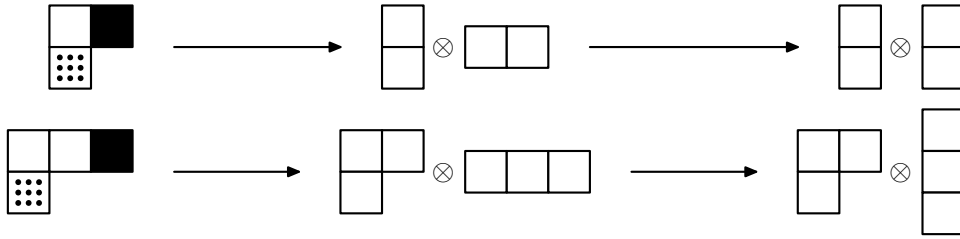
$$(\Sigma_\lambda \dot{V} \otimes \Sigma_{\lambda'} \dot{V} \otimes S^{q_1}\dot{V} \otimes S^{q_2}\dot{V}) \otimes (\Sigma_\mu \dot{V} \otimes \Sigma_{\mu'} \dot{V} \otimes S^{q_2}\dot{V} \otimes S^{q_2}\dot{V})$$

are annihilated, if $l(\lambda) + l(\mu) \neq l(\nu)$. All the other components of the product will appear. This proves the following Theorem. (Recall that T' is the transposed diagram T and \tilde{T}_k is defined before Theorem 4.1.)

Theorem 5.1. *Let \mathcal{T}_1 and \mathcal{T}_2 be colored Young diagrams of two irreducible G -modules from the syzygy algebra. Let T_1 and T_2 be their white parts. Then the image of multiplication consists of all $T = \left(\widetilde{T'_{l(T)}} \right)'_{l(T)}$ such that $T \subset T_1 \otimes T_2$ and $l(T) = l(T_1) + l(T_2)$. (Note that $l(\tilde{T}_m) = l(T)$ for $m \leq l(T)$.)*

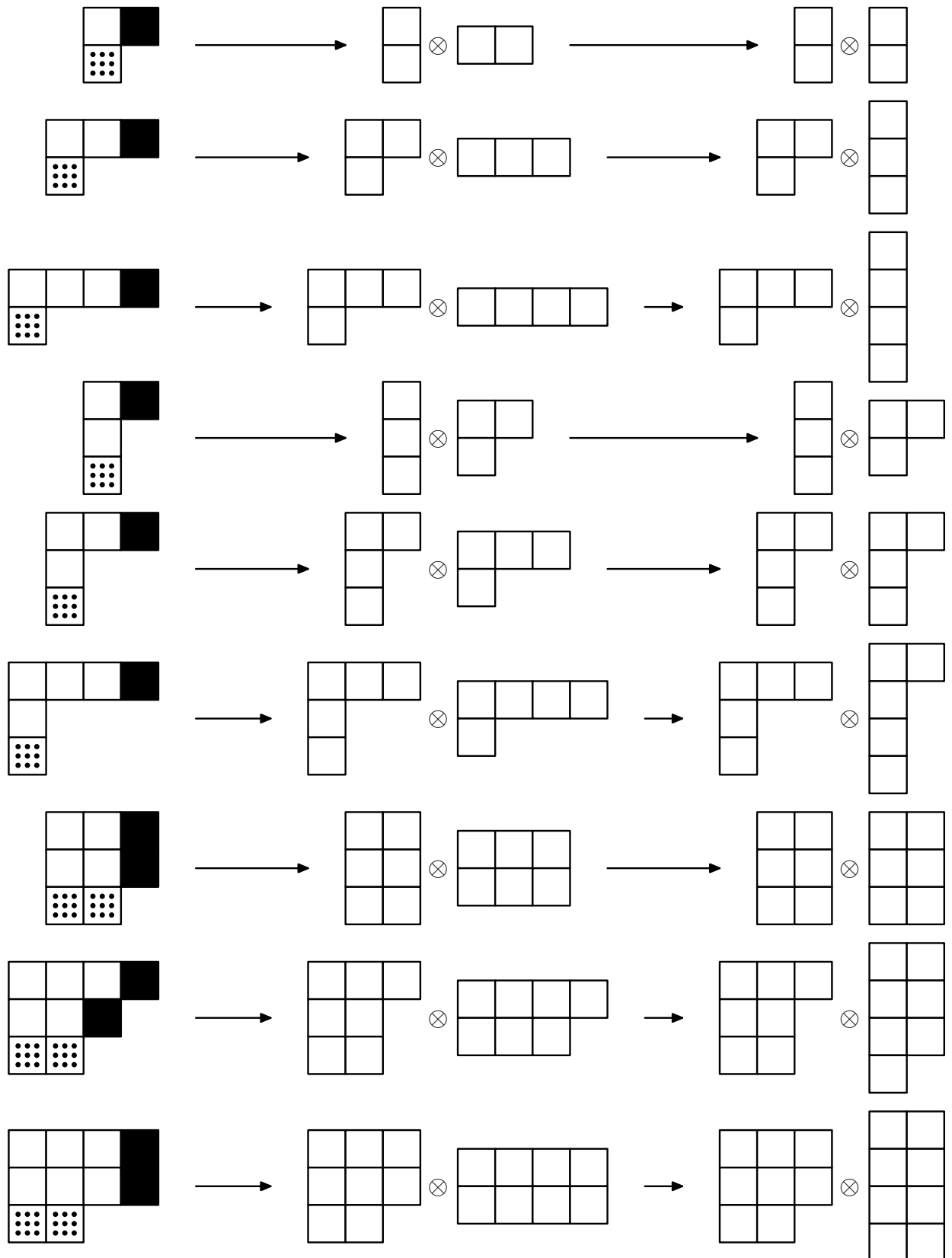
Let us give some examples.

Example 5.2. *Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. There are two diagrams with width no more 2 and height no more 1. The syzygy algebra consists of two $\mathrm{GL}(2) \times \mathrm{GL}(3)$ -modules.*



We get $R_{1,2} = \Lambda^2 \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^3$, $R_{2,3} = \Sigma_{2,1} \mathbb{C}^2 \otimes \Lambda^3 \mathbb{C}^3$, the syzygy algebra has the zero multiplication.

Example 5.3. *Consider the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{11}$. There are nine diagrams with width no more 3 and height no more 2. The syzygy algebra consists of nine $\mathrm{GL}(3) \times \mathrm{GL}(4)$ -modules.*



We get

$$\begin{aligned}
R_{1,2} &= \Lambda^2 \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^4, \\
R_{2,3} &= \Sigma_{2,1} \mathbb{C}^3 \otimes \Lambda^3 \mathbb{C}^4 \oplus \Lambda^3 \mathbb{C}^3 \otimes \Sigma_{1,2} \mathbb{C}^3, \\
R_{3,4} &= \Sigma_{3,1} \mathbb{C}^3 \otimes \Lambda^4 \mathbb{C}^4 \oplus \Sigma_{2,1,1} \mathbb{C}^3 \otimes \Sigma_{2,1,1} \mathbb{C}^4, \\
R_{4,5} &= \Sigma_{3,1,1} \mathbb{C}^3 \otimes \Sigma_{2,1,1,1} \mathbb{C}^4 \\
R_{4,6} &= \Sigma_{2,2,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2} \mathbb{C}^4, \\
R_{5,7} &= \Sigma_{3,2,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2,1} \mathbb{C}^4, \\
R_{6,8} &= \Sigma_{3,3,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2,2} \mathbb{C}^4.
\end{aligned}$$

There are only the following non-zero multiplications:

$$R_{3,4} \times R_{1,2} \rightarrow R_{4,6}, \quad R_{4,5} \times R_{1,2} \rightarrow R_{5,7}, \quad R_{4,5} \times R_{2,3} \rightarrow R_{6,8}.$$

They are induced by natural multiplications:

$$\begin{aligned}
\Sigma_{2,1,1} \mathbb{C}^3 \otimes \Sigma_{2,1,1} \mathbb{C}^4 \times \Lambda^2 \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C} &\rightarrow \Sigma_{2,2,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2} \mathbb{C}^4, \\
\Sigma_{3,1,1} \mathbb{C}^3 \otimes \Sigma_{2,1,1,1} \mathbb{C}^4 \times \Lambda^2 \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C} &\rightarrow \Sigma_{3,2,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2,1} \mathbb{C}^4, \\
\Sigma_{3,1,1} \mathbb{C}^3 \otimes \Sigma_{2,1,1,1} \mathbb{C}^4 \times \Sigma_{2,1} \mathbb{C}^3 \otimes \Lambda^3 \mathbb{C} &\rightarrow \Sigma_{3,3,2} \mathbb{C}^3 \otimes \Sigma_{2,2,2,2} \mathbb{C}^4.
\end{aligned}$$

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